

Anomalies for Nonlocal Dirac Operators

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Abstract. The anomalies of a very general class of non local Dirac operators are computed using the ζ -function definition of the fermionic determinant and an asymmetric version of the Wigner transformation. For the axial anomaly all new terms introduced by the non locality can be brought to the standard minimal Bardeen's form. Some extensions of the present techniques are also commented.

INTRODUCTION

Local field theories provide the commonly accepted setup where the implementation of space-time symmetries becomes rather simple. On the other hand, effective theories are not necessarily local, although an appropriate choice of degrees of freedom can make them almost local [1]. In low energy QCD, light quarks and gluons are dressed by the interaction in a way that the effective theory looks highly nonlocal [2], and a kind of dynamical perturbation theory would be needed [3]. In terms of pions and (heavy) nucleons the theory becomes weakly nonlocal and a chiral perturbation theory becomes of practical interest [4]. In a Dyson-Schwinger setting [5] most information about such a non local theory comes from the constraints imposed by the relevant Ward and Slavnov-Taylor identities [5], perturbation theory to some finite order and hadronic phenomenology. These approaches are necessary if one wants to know, for instance, about the momentum distribution of a quark in a hadron; neither perturbative QCD nor chiral perturbation theory can properly handle this problem. In this regard, anomalies are particularly interesting because their existence is linked to a violation of classical symmetries by high energy regulators although their physical effect is formulated as a low energy theorem. At the one loop level, anomalies for nonlocal models have

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been previously discussed [6–8] for some specific processes like e.g. $\pi^0 \rightarrow 2\gamma$, $\gamma \rightarrow 3\pi$ and $2K \rightarrow 3\pi$. We refer to those works and [9] for further motivation. Rather than computing all specific processes one by one we prove that the new terms generated by the non locality can be subtracted by adding suitable counterterms.

I THE ONE LOOP EFFECTIVE ACTION

Our starting point is the effective action of Dirac fermions in the flat Euclidean space-time endowed with internal degrees of freedom collectively referred to as “flavor”,

$$W(\mathbf{D}) = -\log \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ - \int d^D x \bar{\psi}(x) \mathbf{D} \psi(x) \right\} = -\text{Tr} \log \mathbf{D}. \quad (1)$$

Here, Tr stands for trace over all degrees of freedom and \mathbf{D} is the Dirac operator to be specified below. The definition of the fermion determinant requires some renormalization of the ultraviolet divergences. The (consistent) chiral anomaly is defined as the variation of the effective action under infinitesimal chiral transformations, given by

$$\psi(x) \rightarrow e^{i\beta - i\alpha\gamma_5} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{-i\beta - i\alpha\gamma_5} \quad (2)$$

where $\alpha(x)$ and $\beta(x)$ are Hermitian matrices in flavor space only, regarded as multiplicative operators on the fermionic wave functions. The particular cases $\alpha = 0$ and $\beta = 0$ correspond to vector and axial transformations, respectively. This induces transformations of the Dirac operator

$$\mathbf{D} \rightarrow e^{i\beta - i\alpha\gamma_5} \mathbf{D} e^{-i\beta - i\alpha\gamma_5}, \quad (3)$$

which infinitesimally become

$$\delta \mathbf{D} = \delta_V \mathbf{D} + \delta_A \mathbf{D} = [i\beta, \mathbf{D}] - \{i\alpha\gamma_5, \mathbf{D}\}. \quad (4)$$

Since we will be considering a ζ -function renormalization of W (see below), there will be no vector anomaly,

$$\delta_V W = 0, \quad \delta_A W = \mathcal{A}_A. \quad (5)$$

Correspondingly, the same current conservation formulas valid for the local case can be written here,

$$0 = \int d^D x \langle \bar{\psi}(x) [i\beta, \mathbf{D}] \psi(x) \rangle_Q \quad (6)$$

$$-\mathcal{A}_A = \int d^D x \langle \bar{\psi}(x) \{i\alpha\gamma_5, \mathbf{D}\} \psi(x) \rangle_Q. \quad (7)$$

(The symbol $\langle \rangle_Q$ stands for quantum vacuum expectation value.) In particular the term $\gamma_\mu P_\mu$ in \mathbf{D} in the right-hand side yields, after integration by parts, the divergence of the fermionic vector and axial currents whereas the other terms in \mathbf{D} , local and non local, represent the explicit chiral symmetry breaking due to the external fields. On the other hand, the left-hand side shows the anomalous breaking of the axial current conservation.

The class of Dirac operators to be considered here is

$$\mathbf{D} = \mathbf{D}_L + \mathbf{M}. \quad (8)$$

The term \mathbf{D}_L , the local component of \mathbf{D} , is a standard Dirac operator

$$\mathbf{D}_L = \gamma_\mu P_\mu + \mathbf{Y} \quad (9)$$

We will follow the conventions of [10,11]), $P_\mu = i\partial_\mu$ and \mathbf{Y} is an arbitrary matrix-valued function in flavor and Dirac spaces. \mathbf{Y} is a function of the position operators X_μ , defined by $X_\mu\psi(x) = x_\mu\psi(x)$, so that \mathbf{Y} is a multiplicative operator in the Hilbert space of fermions. The term \mathbf{M} is a purely non local², more precisely bilocal, operator also with arbitrary structure in flavor and Dirac spaces,

$$(\mathbf{Y}\psi)(x) = \mathbf{Y}(x)\psi(x), \quad (\mathbf{M}\psi)(x) = \int d^D y \mathbf{M}(x, y)\psi(y). \quad (10)$$

More restrictive assumptions on \mathbf{M} are spelled out in ref. [11]. For the purpose of doing detailed calculations we will assume that the non local operator \mathbf{M} admits an expansion in inverse powers of P_μ for large P_μ of the form

$$\mathbf{M} = \mathbf{M}_\mu \frac{P_\mu}{P^2} + \mathbf{M}_{\mu\nu} \frac{P_\mu P_\nu}{P^4} + \mathbf{M}_{\mu\nu\rho} \frac{P_\mu P_\nu P_\rho}{P^6} + \dots \quad (11)$$

The coefficients $\mathbf{M}_{\mu_1 \dots \mu_n}$ are multiplicative operators and they are completely symmetric under permutation of indices. For convenience the P_μ has been put at the right³. The better way to obtain the transformation properties of \mathbf{M} is by introducing a family of operators associated to \mathbf{M} as

$$\tilde{\mathbf{M}}(p) = e^{ipX} \mathbf{M} e^{-ipX} \quad (12)$$

where the momentum p_μ is just a constant c-number. Effectively, $\tilde{\mathbf{M}}(p)$ corresponds to make the replacement $P_\mu \rightarrow P_\mu + p_\mu$ in \mathbf{M} . The function $\tilde{\mathbf{M}}(p)$ admits an expansion in inverse powers of p_μ similar to that in eq. (11), namely

²⁾ \mathbf{M} is softer in the ultraviolet sector than any multiplicative operator, that is, the distribution $\mathbf{M}(x, y)$ is less singular than the Dirac delta $\delta(x - y)$.

³⁾ This choice does not exhaust all possible non local operators, but it is realistic since it accommodates the operator product expansion estimate of the quark self-energy $\Sigma(p^2) \sim_{p^2 \rightarrow \infty} (\log p^2)^{d-1}/p^2$ with d the anomalous dimension of the quark condensate $\bar{\psi}\psi$ (see ref. [12])

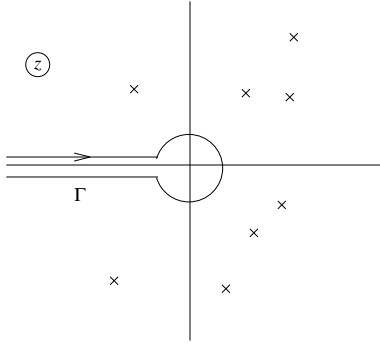


FIGURE 1. The Γ contour in the complex z -plane. Crosses represent isolated eigenvalues of D .

$$\tilde{M}(p) = \tilde{M}_\mu \frac{p_\mu}{p^2} + \tilde{M}_{\mu\nu} \frac{p_\mu p_\nu}{p^4} + \dots \quad (13)$$

We will adopt the ζ -function renormalization prescription combined with an asymmetric Wigner transformation. This method, as well as several of its applications, is presented in great detail in [10]. The ζ -function effective action is given by [13,14]

$$W(D) = -\text{Tr} \log D = -\frac{d}{ds} \text{Tr} (D^s)_{s=0} , \quad (14)$$

where $s = 0$ is to be understood as an analytical extension on s from the ultraviolet convergent region $\text{Re}(s) < -D$. The operator D^s can be obtained from

$$D^s = - \int_{\Gamma} \frac{dz}{2\pi i} \frac{z^s}{D - z} \quad (15)$$

where the integration path Γ starts at $-\infty$, follows the real negative axis, encircles the origin $z = 0$ clockwise and goes back to $-\infty$ (see figure 1). The key point is that for sufficiently negative s there are no ultraviolet divergences and formal operations become justified. By construction, the ζ -function renormalized effective action is invariant under all symmetry transformations associated to similarity transformations of D , thus in particular it is vector gauge invariant. On the other hand, the variable z plays the role of a mass and hence breaks explicitly both chiral and scale invariance.

The operator $(D - z)^{-1}$ can be conveniently expressed by means of an asymmetric version of the Wigner representation [15]. For any operator A , let

$$A(x, p) = \int d^D y e^{iyp} \langle x | A | x - y \rangle = \frac{\langle x | A | p \rangle}{\langle x | p \rangle} \quad (16)$$

be its (asymmetric) Wigner representation. $|p\rangle$ is the momentum eigenstate with $\langle x | p \rangle = e^{-ixp}$. From this definition

$$\langle x|A|x\rangle = \int \frac{d^D p}{(2\pi)^D} A(x, p), \quad \text{Tr} A = \int \frac{d^D x d^D p}{(2\pi)^D} \text{tr} A(x, p), \quad (17)$$

where tr acts on internal and Dirac spinor degrees of freedom only, and the product of two operators satisfy the following formula

$$(AB)(x, p) = \exp(i\partial_p^A \cdot \partial_x^B) A(x, p) B(x, p), \quad (18)$$

where ∂_p^A acts only on the p -dependence in $A(x, p)$ and ∂_x^B on the x -dependence in $B(x, p)$. Let the propagator or resolvent of \mathbf{D} , be $G(z) = (\mathbf{D} - z)^{-1}$ and $G(x, p; z)$ its Wigner representation. Applying Eq.18 to $(\mathbf{D} - z)G = 1$ one obtains

$$G(x, p; z) = \langle x|(\not{p} + \mathbf{D} - z)^{-1}|0\rangle \quad (19)$$

where $|0\rangle$ is the state of zero momentum, $\langle x|0\rangle = 1$. In practice this implies that $i\partial_\mu$ derivates every x dependence at is right, until it annihilates $|0\rangle$, $i\partial_\mu|0\rangle = 0$. This method is very efficient for a derivative expansion for it computes directly, that is non recursively, each of the terms [10].

The definition given for $A(x, p)$ is not gauge covariant because $|p\rangle$ is not. We will consider only local objects of the form $\langle x|f(\mathbf{D})|x\rangle$ as given by the formula

$$\langle x|f(\mathbf{D})|x\rangle = - \int \frac{d^D p}{(2\pi)^D} \int_\Gamma \frac{dz}{2\pi i} f(z) G(x, p; z) \quad (3.7)$$

We will assume that the function $f(z)$ is sufficiently convergent at infinity or else that it can be obtained as a suitable analytical extrapolation from a parametric family $f(z, s)$ in the variable s . In either case the integration over z should be performed in the first place, to yield the Wigner representation of the operator $f(\mathbf{D})$. Afterwards, the p integration is carried out, corresponding to take the diagonal matrix elements of $\langle y|f(\mathbf{D})|x\rangle$, hence restoring gauge covariance. This obviously means that the gauge breaking piece of $G(x, p; z)$ is a total derivative in the momentum variable. Recently, a method has been developped [16] where this total derivative is, by construction, gauge invariant.

II ANOMALIES

Because the chiral transformations are local, both \mathbf{D}_L and \mathbf{M} transform covariantly separately, that is,

$$\delta \mathbf{D}_L = [i\beta, \mathbf{D}_L] - \{i\alpha\gamma_5, \mathbf{D}_L\}, \quad \delta \mathbf{M} = [i\beta, \mathbf{M}] - \{i\alpha\gamma_5, \mathbf{M}\}. \quad (20)$$

Note that the bilocal structure of M implies that local factors at each side of the operator are taken at different points, i.e. $\mathbf{M}(x, x') \rightarrow e^{i\beta(x) - i\alpha(x)\gamma_5} \mathbf{M}(x, x') e^{-i\beta(x') - i\alpha(x')\gamma_5}$.

The two lowest coefficients are given by

$$\tilde{\mathbf{M}}_\mu = \mathbf{M}_\mu, \quad \tilde{\mathbf{M}}_{\mu\nu} = \mathbf{M}_{\mu\nu} + t_{\mu\nu\rho\sigma} \mathbf{M}_\rho P_\sigma, \quad (21)$$

where we have introduced $t_{\mu\nu\rho\sigma} = \delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}$. It should be noted that the coefficients $\tilde{\mathbf{M}}_{\mu_1\dots\mu_n}$ are not multiplicative operators. One useful property of $\tilde{\mathbf{M}}(p)$ is that it transforms covariantly under chiral transformations. Indeed, if $\mathbf{M}_\Omega = \Omega_1 \mathbf{M} \Omega_2$ for two multiplicative operators $\Omega_{1,2}$,

$$\tilde{\mathbf{M}}_\Omega(p) = e^{ipX} \mathbf{M}_\Omega e^{-ipX} = \Omega_1 \tilde{\mathbf{M}}(p) \Omega_2. \quad (22)$$

As a consequence, the coefficients are also chiral covariant

$$\delta \tilde{\mathbf{M}}_{\mu_1\dots\mu_n} = [i\beta, \tilde{\mathbf{M}}_{\mu_1\dots\mu_n}] - \{i\alpha\gamma_5, \tilde{\mathbf{M}}_{\mu_1\dots\mu_n}\}. \quad (23)$$

From here it is immediate to derive the transformation of the original coefficients $\mathbf{M}_{\mu_1\dots\mu_n}$. For the two lowest order coefficients one finds

$$\begin{aligned} \delta \mathbf{M}_\mu &= [i\beta, \mathbf{M}_\mu] - \{i\alpha\gamma_5, \mathbf{M}_\mu\}, \\ \delta \mathbf{M}_{\mu\nu} &= [i\beta, \mathbf{M}_{\mu\nu}] - \{i\alpha\gamma_5, \mathbf{M}_{\mu\nu}\} + t_{\mu\nu\rho\sigma} \mathbf{M}_\rho (\partial_\sigma \beta + \partial_\sigma \alpha \gamma_5). \end{aligned} \quad (24)$$

In general, the variation of each coefficient involves those of lower order.

The scale transformation $\psi(x) \rightarrow e^{-\alpha_S(D-1)/2} \psi(e^{-\alpha_S} x)$ induces the corresponding transformation in \mathbf{D} , namely,

$$\mathbf{Y}(x) \rightarrow e^{-\alpha_S} \mathbf{Y}(e^{-\alpha_S} x), \quad \mathbf{M}_{\mu_1\dots\mu_n}(x) \rightarrow e^{-\alpha_S(n+1)} \mathbf{M}_{\mu_1\dots\mu_n}(e^{-\alpha_S} x). \quad (25)$$

Infinitesimally, it implies

$$\delta_S \mathbf{D} = -\alpha_S (\mathbf{D} - i[X_\mu P_\mu, \mathbf{D}]). \quad (26)$$

A Chiral anomaly

Due to the ζ regularization the chiral anomaly becomes an axial anomaly,

$$\mathcal{A}_A = \text{Tr} (2i\alpha\gamma_5 \mathbf{D}^s)_{s=0}. \quad (27)$$

Using the Wigner transformation technique [10], the anomaly can be written as (a similar expression holds for the effective action)

$$\mathcal{A}_A = - \int \frac{d^D p}{(2\pi)^D} \int_\Gamma \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | 2i\alpha\gamma_5 \frac{1}{\tilde{\mathbf{D}}(p) - z} | 0 \rangle \Big|_{s=0}. \quad (28)$$

Here tr stands for trace over Dirac and flavor degrees of freedom, $|0\rangle$ is the zero momentum state normalized as $\langle x|0\rangle = 1$, thus $P_\mu |0\rangle = \langle 0|P_\mu = 0$. Further

$$\tilde{\mathbf{D}}(p) = e^{ipX} \mathbf{D} e^{-ipX} = \not{p} + \mathbf{D}_L + \tilde{\mathbf{M}}(p). \quad (29)$$

The integration over z should be performed first, since it defines the operator \mathbf{D}^s , then the integral over p which corresponds to take the trace over space-time degrees of freedom and finally s is to be analytically extended to $s = 0$. The simplest way to proceed is to introduce a mass term, i.e., to apply the formula to the Dirac operator $\mathbf{D} + m$ and then make an expansion in powers of $\mathbf{D}_L + \tilde{\mathbf{M}}(p)$, letting $m \rightarrow 0$ at the end. In this way the following expression is derived

$$\mathcal{A}_A = \sum_{N \geq 0} \int \frac{d^D p}{(2\pi)^D} \int_{\Gamma} \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | 2i\alpha\gamma_5 \frac{(\mathbf{D}_L + \tilde{\mathbf{M}}(p)) \left((\not{p} + z - m)(\mathbf{D}_L + \tilde{\mathbf{M}}(p)) \right)^N}{(p^2 + (z - m)^2)^{N+1}} | 0 \rangle \Big|_{s=0, m=0}. \quad (30)$$

Simplification has been achieved by using the cyclic property for the trace in Dirac space.

After an angular average over p_μ , the indicated integrals on p_μ and z can be carried out directly with the integral I_1 given in [10]. The result for the four dimensional chiral anomaly is

$$\begin{aligned} \mathcal{A}_A = & \left\langle 2i\alpha\gamma_5 \left[\frac{1}{2} \mathbf{D}_L^4 + \frac{1}{12} \mathbf{D}_L \{ \gamma_\mu, \mathbf{D}_L \}^2 \mathbf{D}_L + \frac{1}{4} \tilde{\mathbf{M}}_\mu^2 + \frac{1}{4} \{ \mathbf{D}_L, \tilde{\mathbf{M}}_{\mu\mu} \} \right. \right. \\ & \left. \left. + \frac{1}{8} \left(\tilde{\mathbf{M}}_\mu \{ \gamma_\mu, \mathbf{D}_L \} \mathbf{D}_L + \mathbf{D}_L \{ \gamma_\mu, \tilde{\mathbf{M}}_\mu \} \mathbf{D}_L + \mathbf{D}_L \{ \gamma_\mu, \mathbf{D}_L \} \tilde{\mathbf{M}}_\mu \right) \right] \right\rangle \quad (31) \end{aligned}$$

The notation $\langle f \rangle$ stands for $\langle f \rangle = \frac{1}{(4\pi)^{D/2}} \text{tr} \langle 0 | f(X) | 0 \rangle$. Note that, even for non local Dirac operators, the anomaly is a local polynomial of dimension D constructed with P_μ and the external fields \mathbf{Y} and $\mathbf{M}_{\mu_1 \dots \mu_n}$. This is a general property of all anomalies since only ultraviolet divergent terms can contribute to them. The expressions found for the anomaly can be put in a more usual form, in terms of vector and axial fields, scalar fields, etc ⁴, but it is preferable to use compact notation. Since the regularization preserves vector gauge invariance, the axial anomaly is also invariant. In our expression for the anomaly, this is a direct consequence of the operators there being multiplicative. Indeed, any operator f constructed with the gauge covariant blocks \mathbf{D}_L and $\tilde{\mathbf{M}}_{\mu_1 \dots \mu_n}$ is also covariant, i.e., $f \rightarrow \Omega f \Omega^{-1}$. If in addition f is multiplicative $\langle f \rangle$ is invariant. Note that $\langle \rangle$ is not a trace and so the cyclic property does not hold for arbitrary non multiplicative operators.

B Trace anomaly

The corresponding trace anomaly, within the ζ -function method is [10]

⁴) See ref [10] for an explicit expression in the local case and the remarks of ref [11] for the non local case.

$$\mathcal{A}_S = \delta_S W = \alpha_S \text{Tr}(\mathbf{D}^s)_{s=0}. \quad (32)$$

The calculation is entirely similar to that of the axial anomaly, yielding

$$\begin{aligned} \mathcal{A}_S = \alpha_S \bigg\langle & \frac{1}{2} \mathbf{D}_L^4 + \frac{1}{12} (\mathbf{D}_L^2 \{\gamma_\mu, \mathbf{D}_L\}^2 + \{\gamma_\mu, \mathbf{D}_L\} \mathbf{D}_L^2 \{\gamma_\mu, \mathbf{D}_L\} + \{\gamma_\mu, \mathbf{D}_L\}^2 \mathbf{D}_L^2) \\ & + \frac{1}{96} (\{\gamma_\mu, \mathbf{D}_L\}^2 \{\gamma_\nu, \mathbf{D}_L\}^2 + (\{\gamma_\mu, \mathbf{D}_L\} \{\gamma_\nu, \mathbf{D}_L\})^2 + \{\gamma_\mu, \mathbf{D}_L\} \{\gamma_\nu, \mathbf{D}_L\}^2 \{\gamma_\mu, \mathbf{D}_L\}) \\ & + \frac{1}{12} (\gamma_\mu \tilde{\mathbf{M}}_\mu \mathbf{D}_L^2 + \gamma_\mu \mathbf{D}_L^2 \tilde{\mathbf{M}}_\mu + \gamma_\mu \mathbf{D}_L \tilde{\mathbf{M}}_\mu \mathbf{D}_L) \\ & - \frac{1}{24} (\tilde{\mathbf{M}}_\mu \gamma_\mu \mathbf{D}_L^2 + \mathbf{D}_L^2 \gamma_\mu \tilde{\mathbf{M}}_\mu + \mathbf{D}_L \{\gamma_\mu, \tilde{\mathbf{M}}_\mu\} \mathbf{D}_L + \mathbf{D}_L \gamma_\mu \mathbf{D}_L \tilde{\mathbf{M}}_\mu + \tilde{\mathbf{M}}_\mu \mathbf{D}_L \gamma_\mu \mathbf{D}_L) \\ & + \delta_{\mu\nu\alpha\beta} \left(\frac{1}{36} (\gamma_\mu \tilde{\mathbf{M}}_\nu \gamma_\alpha \mathbf{D}_L \gamma_\beta \mathbf{D}_L + \gamma_\mu \mathbf{D}_L \gamma_\alpha \tilde{\mathbf{M}}_\nu \gamma_\beta \mathbf{D}_L + \gamma_\mu \mathbf{D}_L \gamma_\alpha \mathbf{D}_L \gamma_\beta \tilde{\mathbf{M}}_\nu) \right. \\ & \left. + \frac{1}{24} \gamma_\mu \tilde{\mathbf{M}}_\nu \gamma_\alpha \tilde{\mathbf{M}}_\beta + \frac{1}{24} \{\gamma_\alpha \tilde{\mathbf{M}}_{\mu\nu}, \gamma_\beta \mathbf{D}_L\} + \frac{1}{12} \gamma_\mu \tilde{\mathbf{M}}_{\nu\alpha\beta} \right) \bigg\rangle. \quad (33) \end{aligned}$$

Where $\delta_{\mu\nu\alpha\beta} = \delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\alpha\nu}$. The result is again a local polynomial of dimension D in the external fields and their derivatives. Unlike the axial case, the coefficients $\mathbf{M}_{\mu\nu\alpha}$ in four dimensions do contribute to the scale anomaly.

Because scale and chiral transformations commute (in a properly defined sense), the crossed variations $\delta_S \mathcal{A}_{V,A}$ and $\delta_{V,A} \mathcal{A}_S$ coincide and they vanish since the axial anomaly is scale invariant. Thus the scale anomaly must be chiral invariant. The vector gauge invariance of the previous expressions is easy to check noting that the operators inside $\langle \rangle$ are multiplicative. Axial invariance is much more involved in general. In four dimensions it is relatively easy to check that the trace anomaly is axially invariant in the particular case of $\mathbf{M}_\mu = 0$, which defines a class of operators invariant under chiral and scale transformations.

C Counterterms and minimal form of the anomaly

Presumably due to its topological connection [17], the axial anomaly is a very robust quantity. It is not affected by higher order radiative corrections [18], and remains unchanged at finite temperature and density [19]. It gets no contributions from scalar and pseudo scalar fields [20], tensor fields [21,22,10] or internal gauge fields, i.e, transforming homogeneously under gauge transformations [23,24]. In all known cases, the anomaly only affects the imaginary part of the effective action in Euclidean space and only involves vector and axial fields. The counter terms can always be chosen so that the axial anomaly adopts the minimal or Bardeen's form [20]. Not surprisingly, the new terms introduced in the anomaly by the non local component of the Dirac operator are also unessential, that is, they can be removed by adding

a suitable local and polynomial counter term to the effective action. In other words, all new terms can be derived as the axial variation of an action which is a polynomial constructed with the external fields \mathbf{Y} and $\mathbf{M}_{\mu_1 \dots \mu_n}$ and their derivatives. The dimension of the polynomial can be at most D .

The general proof that the anomaly can always be brought to Bardeen's form has already been presented [11]. The actual construction of the counter terms can be done using the method in ref. [10] (see some further details in ref. [27]). One interesting insight in the local case [10] is that the needed counterterms require not only the Dirac operator \mathbf{D} but also its adjoint \mathbf{D}^\dagger not related to the original theory.

The scale anomaly is already minimal. It can be modified by adding polynomial counter terms of dimension smaller than D but this would add terms of the same type to the scale anomaly.

III EXTENSION TO FINITE TEMPERATURE

The Wigner transformation method combined with the ζ -function regularization has been further extended to the finite temperature case in [28,29]. As is well known, in the imaginary time formulation of finite temperature field theory, the field configurations are periodic or antiperiodic functions of the Euclidean time for bosons and fermions respectively and thus the frequency running in the fermion loop takes discrete values only, $\omega_n = \pi(2n + 1)T$ (where T stands for the temperature and n is any integer) which are known as Matsubara frequencies. At finite temperature, the trace of an operator $f(x_0, \mathbf{x}; i\partial_0, i\nabla)$, acting on the Hilbert space of $d + 1$ -dimensional fermions with possible internal degrees of freedom, becomes

$$\text{Tr}(f) = T \sum_n \int \frac{d^d p}{(2\pi)^d} \text{tr} \langle 0 | f(x_0, \mathbf{x}; \omega_n + i\partial_0, \mathbf{p} + i\nabla) | 0 \rangle. \quad (34)$$

This formula generalizes that for zero temperature. Note that f is a periodic function of x_0 , and $|0\rangle$ is the state with zero momentum and energy normalized to $\langle x_0, \mathbf{x} | 0 \rangle = 1$, and thus it is periodic too.

In Ref. [28] this method has been applied to compute the anomalous component of the effective action of two- and four-dimensional fermions at finite temperature in the presence of arbitrary vector and axial gauge fields and scalar and pseudoscalar fields on the chiral circle. The computation is carried out to leading order in a suitable commutator which preserves chiral symmetry. As is well known, at zero temperature the gauge Wess-Zumino-Witten (WZW) action, which saturates the chiral anomaly, is the only leading contribution in the anomalous sector; further terms must be Lorentz and chiral invariant and they vanish identically unless they have more gradients thereby being sub-leading terms. At finite temperature the situation is different since

Lorentz invariance is partially broken and this allows to have new chiral invariant contributions⁵ of the same order as the WZW action. In particular these terms modify the $\pi \rightarrow \gamma\gamma$ amplitude which is no longer determined by the chiral anomaly [30]. The calculation in [28] confirms previous results [31,32] that this amplitude vanishes in a chiral symmetric phase (see also [33,34]).

In Ref. [29] the same technique is applied to the study of 2 + 1-dimensional fermions at finite temperature in the presence of arbitrary background gauge fields. The use of the ζ -function regularization guarantees the gauge invariance of the result under topologically small and large transformations. This has allowed to solve a long standing puzzle, namely, the apparent renormalization of the Chern-Simons coefficient at finite temperature, which has been shown to be a perturbation theory artifact [35]. In Ref. [29] all ultraviolet divergent terms of the effective action, within a strict gradient expansion, have been computed. The result preserves gauge and parity symmetries (up to the standard temperature independent parity anomaly) and display the correct $2\pi i$ multivaluation introduced by the Chern-Simons term. The known exact result for massless fermions [36] is also reproduced.

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⁵) It is well established that the chiral anomaly is temperature independent (see e.g. [19]).

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